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A Sequence of Duals for $Sp(2N)$ Supersymmetric Gauge Theories with Adjoint Matter

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Abstract

We consider supersymmetric $Sp(2N)$ gauge theories with F matter fields in the defining representation, one matter field in the adjoint representation, and no superpotential. We construct a sequence of dual descriptions of this theory using the dualities of Seiberg combined with the "deconfinement" method introduced by Berkooz. Our duals hint at a new non-perturbative phenomenon that seems to be taking place at asymptotically low energies in these theories: for small F some of the degrees of freedom form massless, non-interacting bound states while the theory remains in an interacting non-Abelian Coulomb phase. This phenomenon is the result of strong coupling gauge dynamics in the original description, but has a simple classical origin in the dual descriptions. The methods used for constructing these duals can be generalized to any model involving arbitrary 2-index tensor representations of $Sp(2N)$, $SO(N)$, or $SU(N)$ groups.

1 Introduction

Recently there has been considerable progress in understanding non-perturbative effects in supersymmetric gauge theories (see for example Refs. [1, 2, 3, 4]). In particular, Seiberg [3] has argued convincingly that the low-energy dynamics of supersymmetric $SU(N)$ QCD can be described by a dual gauge theory with different gauge group and matter content. (The $SO(N)$ and $Sp(2N)$ cases were worked out in detail in Refs. [5] and [6], respectively.) Dual descriptions have since been discovered for a wide range of theories; see for example Refs. [7, 8, 9].

One theory that has attracted considerable attention recently is the model with gauge group $SU(N)$, $SO(N)$, or $Sp(2N)$, containing F vector-like "flavors" of matter fields in the defining representation and one matter field, A, in the adjoint representation [10, 11, 12, 13]. Kutasov and Schwimmer [10, 11] have constructed dual descriptions for the $SU(N)$ theory with the addition of a superpotential of the form

$$W = \mathrm{tr}(A^{k+1}). \tag{1.1}$$

The $SO(N)$ and $Sp(2N)$ analogs of this model were worked out in Ref. [12] (see also Ref. [16]). It was found that the size of the dual gauge group depends on k, and becomes infinitely large as $k \to \infty$. More recently, Kutasov, Schwimmer, and Seiberg [13] have obtained impressive detailed evidence for the validity of the duality presented in Refs. [10, 11].

The dynamics of the theory without a superpotential is still not well understood, although there are some hints coming from analyzing the theory with various superpotentials added. Using the dual descriptions of Refs. [10, 11, 13] it can be shown that in the presence of the superpotential Eq. (1.1), the theory is at an interacting superconformal fixed point for $2N/(2k-1) < F < 2N$. Taking the limit $k \to \infty$ (which makes the superpotential arbitrarily flat at the origin) one can argue that the low-energy dynamics of the theory without a superpotential is described by an interacting superconformal fixed point for $0 < F < 2N$ [13]. This suggestion is in accord with the results of Ref. [14], which studied the theory with the addition of a superpotential

$$W = \lambda \hat{Q} A Q. \tag{1.2}$$

where Q and $\hat{Q}$ are the fundamental and antifundamental matter fields, respectively. For finite λ the theory is smoothly connected to $N = 2$ supersymmetric QCD, which is known to be in an Abelian Coulomb phase [15, 14]. When the limit $\lambda \to 0$ is taken, it is found that singularities appear in the low-energy effective action [14], as expected if the theory is entering a non-Abelian Coulomb phase, and the requisite gauge bosons are becoming massless. In this paper, we further investigate the conjecture that the theory without a superpotential is at an interacting superconformal fixed point for all $0 < F < 2N$.

We will study the $Sp(2N)$ version of this theory. Its properties are expected to be very similar to the $SU(N)$ case, but it is easier to analyze because the invariants of $Sp(2N)$ are simpler. Assuming that the theory is at a non-trivial superconformal fixed point, we show that some massless degrees of freedom must decouple from the superconformal fixed point theory for $F < F_0$, where $F_0 \geq \frac{1}{2}(N+1)$. We describe the various possibilities for which operators decouple in the infrared.

We then construct a series of dual descriptions that suggest that gauge invariant operators of the form QA^kQ, $k = 0, 1, 2, \ldots$ sequentially decouple as F is reduced. The dual descriptions are constructed by generalizing the "deconfinement" method introduced by Berkooz [8, 9]. The first dual description is a theory with gauge group $Sp \times SO$, with matter fields in the fundamental and adjoint representations, and a superpotential. We then iterate the process, applying the deconfinement method to SO and Sp groups to obtain more complicated dual descriptions.[1]

This paper is organized as follows. In Section 2 we introduce the theory under investigation and consider some of the possible scenarios for the infrared physics. In Section 3 we describe the construction of the first dual, and in Section 4 we iterate the construction to obtain additional duals that we then use to speculate about the infrared spectrum. In the appendices, we perform some consistency checks on these dual descriptions and show how the deconfinement method can be generalized to arbitrary 2-index representations.

2 The Model

The theory we wish to study has gauge group $Sp(2N)$ with $2F$ matter fields Q in the defining representation, and one matter field A in the adjoint (symmetric second rank tensor) representation. It has the anomaly-free global symmetry $SU(2F) \times U(1) \times U(1)_R$. The field content (with global charges) is given in table 1.

field	$Sp(2N)$	$SU(2F)$	$U(1)$	$U(1)_R$
Q	$\square$	$\square$	$\frac{N+1}{F}$	1
A	$\square\!\square$	$\mathbf{1}$	-1	0

Table 1: Field content of the theory.

The moduli space can be parameterized by the holomorphic gauge-invariant polynomials in the matter fields (the "chiral ring").[2] In the present case, these are generated by

$$T_k \equiv \operatorname{tr} A^{2k}, \quad k = 1, 2, \ldots \tag{2.1}$$
$$M_k \equiv QA^kQ, \quad k = 0, 1, \ldots \tag{2.2}$$

[1] A similar series of duals for an SU theory with an antisymmetric tensor was noted in [9].
[2] This connection has been known for some time; for a proof, see Ref. [17].

In Ref. [12] (following Refs. [10, 11]) this theory is studied with the addition of a superpotential

$$W = \mathrm{tr}\, A^{2k}, \tag{2.3}$$

and different dual descriptions are constructed for each value of $k \geq 1$. The dual gauge group is $Sp(2\widetilde{N})$, where $\widetilde{N} = (2k+1)F - (N+2)$.

We will consider the theory with no superpotential. We will often take advantage of the simplifications of the large N limit (with N/F held fixed), but we believe that most of these results remain valid for $N \sim 1$. The exact β-function of the theory satisfies [18]

$$\beta \propto 2(N+1) - F(1-\gamma_{QQ}) + (N+1)\gamma_{AA}, \tag{2.4}$$

where γ_{QQ} and γ_{AA} are the anomalous dimensions of the operators QQ and $\mathrm{tr}\, A^2$, respectively. The theory is infrared free for $F \geq 2(N+1)$. For F just below $2(N+1)$ the one-loop coefficient of the β-function is small and positive while the two-loop coefficient is negative of order N^2. Arguments due to Banks and Zaks [19] show that the gauge coupling has a non-trivial perturbative fixed point in the infrared (at least in the large-N limit). At the fixed point we can calculate anomalous dimensions in perturbation theory:

$$\begin{aligned} \gamma_{AA} &= -\frac{\alpha_*}{\pi}(N+1) + O(\alpha_*^2) \\ \gamma_{QQ} &= -\frac{\alpha_*}{2\pi}(N+\frac{1}{2}) + O(\alpha_*^2) \end{aligned} \tag{2.5}$$

where α_* is the gauge coupling at the fixed point, given by

$$\frac{\alpha_*}{\pi}(N+1) = \frac{\epsilon}{2} + O(\epsilon^2), \qquad \epsilon \equiv 2 - \frac{F}{N+1}. \tag{2.6}$$

The superconformal algebra in the infrared includes an anomaly-free R symmetry whose charges, R_{SC}, are related to the dimensions of chiral operators $\mathcal{O}$ by [20]

$$\dim(\mathcal{O}) = \frac{3}{2} R_{\mathrm{SC}}(\mathcal{O}). \tag{2.7}$$

The R_{SC} charges must be a linear combinations of the anomaly-free $U(1)$ symmetries of the ultraviolet:

$$R_{\mathrm{SC}} = R - bU \tag{2.8}$$

where R and U denote charges under the $U(1)_R$ and $U(1)$, respectively. The coefficient b is a function of F and N that we would like to determine, since b determines the scaling dimensions of the operators in the superconformal algebra. Near the Banks–Zaks fixed point

$$b = \frac{2}{3} - \frac{\epsilon}{6} + O(\epsilon^2). \tag{2.9}$$

As we decrease F away from $2(N+1)$ the gauge coupling at the fixed point increases; eventually perturbation theory breaks down and we do not know how to determine b. However, assuming that the theory is at a conformal fixed point in the infrared we can find limits on b by using the fact that the dimensions of gauge invariant chiral operators satisfy the unitarity bound [20] $\dim(\mathcal{O}) \geq 1$. The bound is saturated for free fields. In Fig. 1 we show the values of b (as a function of F) for which the operators in the chiral ring saturate the bound:

$$\dim(T_k) = 3kb = 1, \qquad \dim(M_k) = 3 - 3b\left(\frac{N+1}{F} - \frac{k}{2}\right) = 1. \tag{2.10}$$

Assuming that the gauge coupling is at a fixed point in the infrared for $0 < F < 2(N+1)$, for large N we can imagine three qualitatively different scenarios (see Fig. 1):

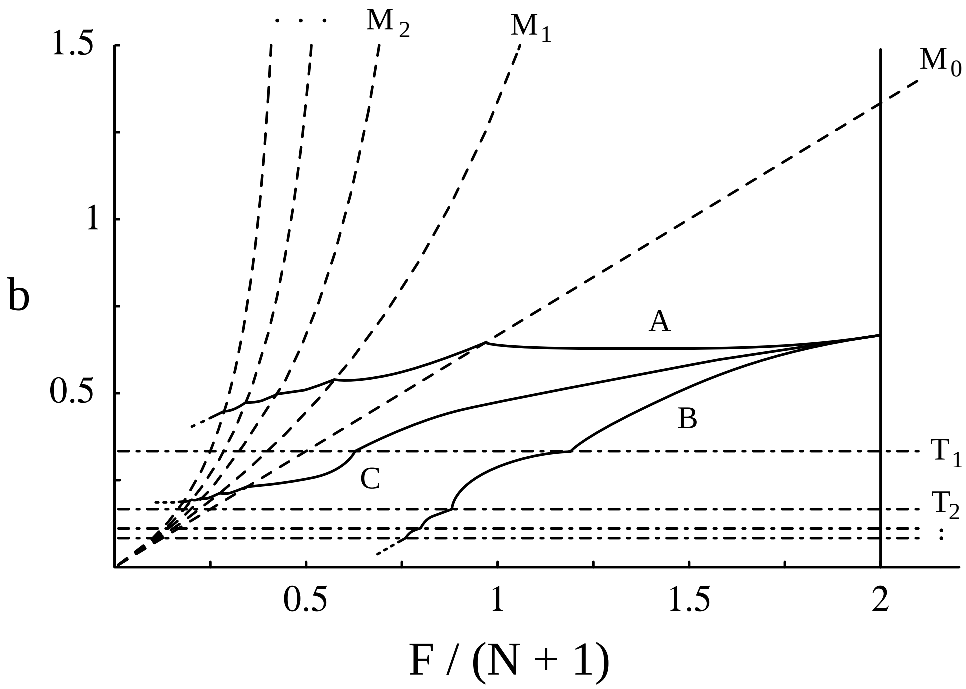

Figure 1: The scenarios A, B, and C for the behavior of the coefficient b that determines the superconformal $R_{\rm SC}$ charges as a function of $F/(N+1)$, for large, fixed N. The curves meet at $F = 2(N+1)$, where the position and slope can be calculated in perturbation theory. For F near zero it presumably does not make sense to plot b as a continuous function. The lines labeled $M_0, M_1, \ldots$ and $T_1, T_2, \ldots$ indicate the region where the corresponding operators have scaling dimensions of a free field.

A. b successively crosses the lines corresponding to $M_0, M_1, \ldots$ as F is reduced. At the value $F = F_0$ where b crosses the M_0 line, the dimension of M_0 as calculated using Eq. (2.8) violates the unitarity bound. This implies that M_0 is a free field (with $\dim(M_0) = 1$) for $F \leq F_0$. The theory can still be at an interacting superconformal fixed point for $F < F_0$, since there is an accidental R_0 symmetry in the infrared under which only the free field transforms. This R_0 symmetry redefines the superconformal R charge of M_0 and allows the dimension of M_0 to stay one. The other fields do not transform under the accidental R_0 symmetry, and therefore the dimensions of all other operators in the theory are still given by Eq. (2.8). We will therefore assume that the the theory remains at a superconformal fixed point. Since the number of degrees of freedom is of order N^2, and we are changing the number of degrees of freedom by order 1 in crossing the M_0 line, it is plausible that b is continuous across the line as shown in Fig. 1. Analogous effects occur as b crosses the lines corresponding to the operators $M_1, M_2, \ldots$.

B. b crosses the lines corresponding to $T_1, T_2, \ldots$ as F is reduced. In this scenario the operators T_k sequentially decouple in a similar fashion as in scenario A above.

C. b crosses lines corresponding to both types of operators M_k and T_k.

Without further information we cannot decide which of these scenarios is correct. In the next sections we will construct dual descriptions that we will use to argue in favor of the first scenario. The operators M_k appear as fundamental fields in the duals and there appear to be values of F and N where they are free fields.

3 Construction of the First Dual

We can find another supersymmetric gauge theory that has the same low-energy dynamics using the "deconfinement" method of Berkooz [8, 9].(In fact this method is quite general and can be used to write a dual description of almost any supersymmetric gauge theory. See Appendix B.) The idea is to replace the adjoint by a composite "meson" operator of a strongly-interacting $SO(N')$ group:

$$A^{ab} \to x^{aa'} x^{ba'}, \tag{3.1}$$

where a, b are $Sp(2N)$ indices and a' is an $SO(N')$ index. N' can be chosen so that $SO(N')$ confines without chiral symmetry breaking [2, 5].[3] The problem with a straightforward application of this idea is that the $Sp(2N) \times SO(N')$ theory has one less anomaly-free $U(1)$ symmetry, since there is an additional constraint that

[3] The branch of the $SO(N')$ theory that generates a dynamical superpotential is eliminated because it has no vacuum.

the $U(1)$ symmetries have no $SO(N')$ anomaly. This problem can be circumvented by introducing additional fields that transform under $SO(N')$ and adding terms to the superpotential in the deconfined description. The matter content of the model that accomplishes this is displayed in Table 4. The superpotential in the deconfined description is

$$W = x_1 p_1 p_2 + p_1 p_1 p_3. \tag{3.2}$$

(We have set the coefficients of the superpotential to +1 by rescaling the fields.) The purpose of the superpotential is to give masses to the the unwanted "meson" states $(x_1 p_1)$ and $(p_1 p_1)$ that appear when the $SO(N')$ group confines.[4] With this matter content, we take $N' = 2N + 5$. Note that the $U(1)$ charges are uniquely determined by the following constraints: they agree with the $U(1)$ charges in the original theory, they are anomaly free, and the superpotential is invariant. We explicitly check that all anomalies of the original theory match those of the dual in appendix A.

field	$Sp(2N)$	$SO(2N+5)$	$SU(2F)$	$U(1)$	$U(1)_R$
Q	$\square$	$\mathbf{1}$	$\square$	$\frac{N+1}{F}$	1
x_1	$\square$	$\square$	$\mathbf{1}$	$-\frac{1}{2}$	0
p_1	$\mathbf{1}$	$\square$	$\mathbf{1}$	N	-2
p_2	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2} - N$	4
p_3	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-2N$	6

Table 2: Field content of the first "deconfined" theory.

We can now use the known dual description of $Sp(2N)$ gauge theory with fundamentals [3, 6] to write a dual description of this theory in terms of a gauge theory with gauge group $Sp(2F+2) \times SO(2N+5)$. The field content is given in Table 3; the dual has superpotential

$$\begin{aligned} W &= M_0 \widetilde{Q}\widetilde{Q} + A_1 \widetilde{x}_1 \widetilde{x}_1 + m_1 \widetilde{Q}\widetilde{x}_1 + m_2 \widetilde{Q}\widetilde{p}_2 + m_3 \widetilde{x}_1 \widetilde{p}_2 \\ &\quad + m_3 p_1 + p_1 p_1 p_3. \end{aligned} \tag{3.3}$$

We can integrate out the massive fields m_3 and p_1, leaving the superpotential (after field rescaling)

$$W = M_0 \widetilde{Q}\widetilde{Q} + A_1 \widetilde{x}_1 \widetilde{x}_1 + m_1 \widetilde{Q}\widetilde{x}_1 + m_2 \widetilde{Q}\widetilde{p}_2 + (\widetilde{x}_1 \widetilde{p}_2)(\widetilde{x}_1 \widetilde{p}_2) p_3. \tag{3.4}$$

The anomaly matching is guaranteed to work by the anomaly matching of the Sp duality used in the construction. The gauge-invariant chiral operators of the original

[4]In generalizations of this deconfinement method to other groups, the confining gauge group generates a dynamical superpotential for the composite fields. In these cases, the analogs of the superpotential terms discussed above also serve to eliminate this superpotential via their equations of motion. See Appendix B.

theory map into the dual description as follows:

$$\begin{aligned} \operatorname{tr} A^{2k} &\rightarrow \operatorname{tr} A_1^{2k}, \quad k = 1, 2 \ldots \\ QQ &\rightarrow M_0, \\ QA^kQ &\rightarrow m_1 A_1^{k-1} m_1, \quad k = 1, 2, \ldots . \end{aligned} \tag{3.5}$$

Note that the composite operator $M_0 \equiv QQ$ of the original theory is a fundamental field in the dual description. This is similar to Seiberg's dual description of supersymmetric QCD [3]. However, the theory we are considering here has a much more complicated structure; there are many operators in the chiral ring that do not map onto elementary fields in the first dual. We will see in the following section that the operators $M_k \equiv QA^kQ$ map onto fundamental fields in the n-th dual for $n > k$, while operators of the form $T_k \equiv \operatorname{tr} A^{2k}$ never appear as fundamental fields in our duals.

field	$Sp(2F+2)$	$SO(2N+5)$	$SU(2F)$	$U(1)$	$U(1)_R$
$\widetilde{Q}$	$\square$	$\mathbf{1}$	$\overline{\square}$	$-\frac{N+1}{F}$	0
M_0	$\mathbf{1}$	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$2\frac{N+1}{F}$	2
$\widetilde{x}_1$	$\square$	$\square$	$\mathbf{1}$	$\frac{1}{2}$	1
A_1	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	-1	0
m_1	$\mathbf{1}$	$\square$	$\square$	$\frac{N+1}{F} - \frac{1}{2}$	1
m_2	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\frac{N+1}{F} + \frac{1}{2} - N$	5
$\widetilde{p}_2$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{1}{2} + N$	-3
p_3	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-2N$	6
p_1	$\mathbf{1}$	$\square$	$\mathbf{1}$	N	-2
m_3	$\mathbf{1}$	$\square$	$\mathbf{1}$	$-N$	4

Table 3: Field content of the first dual description.

What dynamical information can we obtain from this dual? One might have hoped that our dual descriptions would be free in the infrared for some range of N and F, so that our dual gives a weakly-coupled description of the low-energy physics. (This happens in the dual description of supersymmetric QCD for $N + 1 < F < \frac{3}{2}N$.) At one loop, the SO group is infrared free for $F \geq N + 1$ and the Sp group is infrared free for $F \leq \frac{1}{2}(N-3)$. Therefore, this dual description is never completely free in the infrared. (A similar situation holds in all of our duals.) This is not surprising, since we expect that the theory is at an interacting superconformal fixed point, and such a theory cannot have a dual description that is free in the infrared.

In the range $N + 1 \leq F < 2N$ the SO gauge group is infrared free (at one loop), and the Sp gauge group has the right number of colors and flavors (again at one loop) to be in the "conformal window" where there is an interacting superconformal

fixed point [3, 6]. The one-loop calculation amounts to neglecting the contributions of the anomalous dimensions to the β function in Eq. (2.4); this gives qualitatively wrong results when the anomalous dimensions are large due to relevant interactions in the superpotential or strong gauge interactions. An example of such a situation is our dual for $F \lesssim 2N$. The (incorrect) one-loop calculation suggests that the SO interactions can be ignored in the infrared. Ignoring SO we would find that the Sp gauge group is near the end of its "conformal window" and the anomalous dimension of the Sp-gauge invariant operator $\tilde{x}_1\tilde{x}_1$ would be near 1 in analogy to supersymmetric QCD. Then the relevant term in the superpotential $A_1\tilde{x}_1\tilde{x}_1$ would force the dimension of A_1 to be near 2. But by the operator map, tr A_1^{2k} corresponds to the operator tr A^{2k} in the original description, and the dimension of A is close to 1 at the Banks–Zaks fixed point, thus contradicting our naïve interpretation of the dual.

This example illustrates that a one-loop calculation of β functions is not reliable. The reason is that relevant superpotential couplings and strong gauge groups contribute large anomalous dimensions to the fields of the theory which cannot be ignored when calculating β functions. The gauge group that was naïvely believed to be free may even be rendered relevant in the infrared via the anomalous dimensions. This may occur whenever there are fields transforming under both groups, or when there are relevant superpotential couplings. Therefore, one must be very careful in trying to draw physical conclusions from a dual description that is not weakly coupled.

Another obstacle to extracting low-energy physics from this dual is that in the deconfined theory, we introduced massive degrees of freedom in order to cancel anomalies. Once we pass from the deconfined description to the dual description, the fact that these degrees of freedom are irrelevant in the infrared is no longer evident.

While keeping these points in mind, we will nonetheless use this dual and its generalizations below to argue that the operators M_k are free fields for sufficiently small F. In Appendix A we will also perform some consistency checks on this dual description. These help convince us that the dual description is correct, but by themselves they do not give us dynamical information that we do not already know from the original description of the theory.

4 More Dual Descriptions

We can obtain additional dual descriptions by applying the deconfinement method again, this time to the adjoint of the SO group in the first dual. To this end, we introduce a confining Sp group that forms a composite meson with the same quantum numbers as the SO adjoint A_1. (This is precisely the version of deconfinement discussed in Ref. [9].) The field content is given in Table 4, and the superpotential is

$$W = M_0\widetilde{Q}\widetilde{Q} + (x_2x_2)\tilde{x}_1\tilde{x}_1 + m_1\widetilde{Q}\tilde{x}_1 + m_2\widetilde{Q}\tilde{p}_2 + (\tilde{x}_1\tilde{p}_2)(\tilde{x}_1\tilde{p}_2)p_3 + x_2r_1r_2. \qquad (4.1)$$

We can now use the known dual description of SO gauge theory with fundamentals [3, 5] to write a dual description of this theory in terms of a theory with gauge group

field	$Sp(2F+2)$	$SO(2N+5)$	$Sp(2N+2)$	$SU(2F)$	$U(1)$	$U(1)_R$
$\widetilde{Q}$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$	$-\frac{N+1}{F}$	0
M_0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$2\frac{N+1}{F}$	2
$\widetilde{x}_1$	$\square$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\frac{1}{2}$	1
x_2	$\mathbf{1}$	$\square$	$\square$	$\mathbf{1}$	$-\frac{1}{2}$	0
r_1	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$N+\frac{5}{2}$	2
r_2	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$-N-2$	0
m_1	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\square$	$\frac{N+1}{F}-\frac{1}{2}$	1
m_2	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\frac{N+1}{F}+\frac{1}{2}-N$	5
$\widetilde{p}_2$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{1}{2}+N$	-3
p_3	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-2N$	6

Table 4: Field content of the second "deconfined" description.

$Sp(2F+2)\times SO(4F+4)\times Sp(2N+2)$. Some of the fields are massive and can be integrated out. The field content of the resulting theory is given in Table 5, and the superpotential is

$$\begin{aligned} W &= M_0(\tilde{\tilde{x}}_1\widetilde{m}_1)(\tilde{\tilde{x}}_1\widetilde{m}_1) + (\tilde{\tilde{x}}_1\tilde{x}_2)(\tilde{\tilde{x}}_1\tilde{x}_2) + m_2\tilde{p}_2(\tilde{\tilde{x}}_1\widetilde{m}_1) + n_1\tilde{p}_2^2 p_3 \\ &\quad + n_1\tilde{\tilde{x}}_1\tilde{\tilde{x}}_1 + A_2\tilde{x}_2\tilde{x}_2 + M_1\widetilde{m}_1\widetilde{m}_1 + n_3\tilde{r}_2\tilde{r}_2 \\ &\quad + n_2\tilde{x}_2\widetilde{m}_1 + n_4\tilde{\tilde{x}}_1\tilde{r}_2 + n_5\widetilde{m}_1\tilde{r}_2. \end{aligned} \tag{4.2}$$

The gauge-invariant chiral operators of the original theory map into the second dual description as follows:

$$\begin{array}{ccccl} \mathrm{tr}\,A^{2k} & \to & \mathrm{tr}\,A_1^{2k} & \to & \mathrm{tr}\,A_2^{2k}, \quad k=1,2,\ldots \\ QQ & \to & M_0 & \to & M_0, \\ QAQ & \to & m_1m_1 & \to & M_1, \\ QA^kQ & \to & m_1A_1^{k-1}m_1 & \to & n_2A_2^{k-2}n_2, \quad k=2,3,\ldots. \end{array} \tag{4.3}$$

As already stated in the previous section, both the composite operators $M_0=QQ$ and $M_1=QAQ$ of the original theory are fundamental fields in this description. Note that M_0 only interacts via the superpotential term $M_0(\tilde{\tilde{x}}_1\widetilde{m}_1)(\tilde{\tilde{x}}_1\widetilde{m}_1)$, which has canonical dimension 5. If we could identify a range of F where this operator is irrelevant in the infrared, we would have shown that M_0 is free in that range. For example, this would be the case if either one of the two gauge groups $Sp(2F+2)$ or $SO(4F+4)$ were infrared free. To see this, suppose that $SO(4F+4)$ is infrared free. (At one loop, we would naively conclude that this is the case for $F\le\frac{1}{4}(N-1)$.) Then the superpotential term involving M_0 can be written as the product of operators

field	$Sp(2F+2)$	$SO(4F+4)$	$Sp(2N+2)$	$SU(2F)$	$U(1)$	$U(1)_R$
M_0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$2\frac{N+1}{F}$	2
$\tilde{\tilde{x}}_1$	$\square$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{1}{2}$	0
n_1	$\square\!\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	1	2
$\tilde{x}_2$	$\mathbf{1}$	$\square$	$\square$	$\mathbf{1}$	$\frac{1}{2}$	1
A_2	$\mathbf{1}$	$\mathbf{1}$	$\square\!\square$	$\mathbf{1}$	-1	0
$\widetilde{m}_1$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\overline{\square}$	$\frac{1}{2} - \frac{N+1}{F}$	0
M_1	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\square\!\square$	$2\frac{N+1}{F} - 1$	2
n_2	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\square$	$\frac{N+1}{F} - 1$	1
n_3	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-2N-4$	0
n_4	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-N-\frac{3}{2}$	1
n_5	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\frac{N+1}{F} - \frac{5}{2} - N$	1
m_2	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\frac{N+1}{F} + \frac{1}{2} - N$	5
$\tilde{p}_2$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{1}{2} + N$	-3
p_3	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-2N$	6
$\tilde{r}_2$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$N+2$	1

Table 5: Field content of the second dual description after integrating out massive fields.

M_0, $\tilde{\tilde{x}}_1\tilde{\tilde{x}}_1$, and $\widetilde{m}_1$ that are gauge-invariant under all "active" gauge groups. These operators must each have dimension at least 1, so the superpotential term involving M_0 has dimension at least 4, and M_0 is free in the infrared. This corresponds to the scenario A of Section 2. Of course, the superpotential and the other gauge interactions do affect the range of N and F for which the gauge groups are infrared free. Nonetheless, because we know that *some* operators must become free, we interpret this feature of our dual descriptions as suggesting that scenario A is in fact correct.

Note that in the original description, the decoupling of the field M_0 from the superconformal algebra is a non-perturbative phenomenon. In the dual description (provided we are interpreting it correctly), it simply corresponds to the fact that M_0 couples only through a term in the superpotential with high dimension, and the fact that M_0 is free is a simple classical effect. In this sense, our dual descriptions give a weakly-coupled description of a strong coupling phenomenon in the original theory.

This feature also appears in the duals constructed by Kutasov and Schwimmer [10, 11, 13] for the theory with a superpotential Eq. (1.1): while the T_k never appear as fundamental fields in their duals the M_k do, and they only couple through terms in the superpotential with high canonical dimensions. In the range of F where the dual gauge group is free these terms are irrelevant, and the M_k become free fields.

One can continue constructing duals in this fashion. We have constructed the third dual, and we summarize the matter content of the original theory and the first three duals in Table 6.

dual	gauge group	matter
	$Sp(2N)$	$2F+\mathbf{A}$
1	$Sp(2F+2)$	$2F+2N+6$
	$SO(2N+5)$	$4F+2+\mathbf{A}$
2	$Sp(2F+2)$	$4F+6+\mathbf{A}$
	$SO(4F+4)$	$4F+2N+5$
	$Sp(2N+2)$	$6F+4+\mathbf{A}$
3	$Sp(2F+2)$	$4F+6+\mathbf{A}$
	$SO(4F+4)$	$8F+10+\mathbf{A}$
	$Sp(6F+6)$	$6F+2N+12$
	$SO(2N+7)$	$8F+6+\mathbf{A}$

Table 6: Matter content of all the first three duals. **A** indicates an adjoint.

A simple pattern emerges in these duals: in the n-th dual has $n+1$ gauge groups with the following operator maps:

$$\begin{array}{rcl} \operatorname{tr} A^{2k} & \to & \operatorname{tr} A_n^{2k}, \quad k=1,2,\dots \\ QA^kQ & \to & M_k \quad k=1,\dots,n-1. \end{array} \tag{4.4}$$

The fields M_k for $k < n-1$ only interact via terms in the superpotential that have large canonical dimensions. It is therefore plausible that for sufficiently small F such terms are irrelevant in the infrared. (As above, one can show that this is the case provided that at least one of the gauge groups is infrared free.) As discussed above, this provides evidence that the operators M_k successively become free as F is reduced.

5 Conclusions

We have constructed the first three of an infinite sequence of dual descriptions of an $Sp(2N)$ supersymmetric gauge theory with matter fields transforming as an adjoint and F flavors, with no superpotential. In the n-th dual description, the operators $M_k \equiv QA^kQ$ appear as fundamental fields for $k < n$, and the M_k couple only through superpotential interactions that have large canonical dimensions and are likely to be irrelevant in the infrared. This supports the scenario that the the operators $M_0, M_1, \ldots$ sequentially become free massless fields in the infrared as F is reduced from the asymptotic freedom limit $F = 2(N+1)$. In the original theory this picture arises from nonperturbative quantum effects, while the dual descriptions give a simple classical description of the same physics.

It would be very important to understand for which ranges of N and F the various gauge groups of our duals are weakly coupled, so that our results could be put on firmer ground. Unfortunately, this appears to be a difficult problem, partly due to the interplay of the various different gauge groups, but also because of the additional massive degrees of freedom that we had to introduce in the "deconfinement" in order to match $U(1)$'s.

The extension of these results to SU gauge theories is straightforward using the "deconfinement modules" discussed in Appendix B.

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A Consistency Checks

In this appendix, we consider some consistency checks on the dual descriptions constructed in the main text.

A.1 Anomaly matching

The dual descriptions were derived from known dualities, and so the anomalies are guaranteed to match. For completeness and to check our algebra, we explicitly computed the anomalies in the original description and all of the the dual descriptions discussed above, with the following results:

$$\begin{aligned} SU(2F)^3 &: \; 2N \\ U(1)_R SU(2F)^2 &: \; 0 \\ U(1)_R &: \; 0 \\ U(1)_R^3 &: \; 0 \\ U(1)SU(2F)^2 &: \; \frac{2N(N+1)}{F} \\ U(1) &: \; N(2N+3) \\ U(1)^3 &: \; \frac{4N(N+1)^3}{F} - N(2N+1) \\ U(1)U(1)_R^2 &: \; -N(2N+1) \\ U(1)^2 U(1)_R &: \; -N(2N+1) \end{aligned} \tag{A.1}$$

A.2 Integrating out flavors

Another important check is to add a mass term for some of the Q's and see that this gives consistent results in the original and dual descriptions. Consider adding a superpotential that gives masses to two of the quarks in the original theory:

$$\delta W = m Q_{2F-1} Q_{2F}. \tag{A.2}$$

In the first dual this is mapped to

$$\delta W \to m\left((M_0)_{2F-1,2F} - (M_0)_{2F,2F-1}\right). \tag{A.3}$$

The M_0 equation of motion then requires $\widetilde{Q}_{2F-1}, \widetilde{Q}_{2F}$ to have vacuum expectation values, breaking $Sp(2F+2) \to Sp(2F)$. Also, some of the components of the other fields become massive, and we find that the low-energy theory is precisely the first dual description of a theory with $F-1$ flavors, as required for consistency. We see the familiar pattern that integrating out a flavor in the original theory corresponds to spontaneously breaking the gauge group in the dual description.

In the second dual, the discussion is somewhat more complicated. The mass term in the original theory again maps to $m(M_0)_{2F-1,2F}$. The equations of motion require vacuum expectation values for $\widetilde{\widetilde{x}}_1$ and $\widetilde{m}_1$, which break $Sp(2F+2) \to Sp(2F)$ and $SO(4F+4) \to SO(4F)$. Again, some of the components become massive, and one can show that the resulting low-energy theory is precisely the second dual for $F-1$ flavors.

Another potential check on our dual descriptions would be to add a mass term for the adjoint field in the superpotential. However, in our duals this yields a theory that is strongly coupled for all values of N and F, so it does not provide an additional consistency check.

A.3 Moduli space

We can also check that the moduli spaces are the same in the original and the dual descriptions. For example, consider a direction in moduli space corresponding to $\langle A \rangle \neq 0$, $\langle Q \rangle = 0$. Imposing the D-flatness condition, the simplest possibility is

$$\langle A \rangle = \begin{pmatrix} a\sigma_3 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}. \tag{A.4}$$

This breaks the gauge symmetry $Sp(2N) \to Sp(2N-2) \times U(1)$, and the massless fields transform under the unbroken gauge symmetry as $2F$ fundamentals, an adjoint, and $4F+1$ singlets. The fundamentals and adjoint are neutral under the $U(1)$ gauge symmetry, so the theory breaks up into three decoupled sectors in the far infrared: the first is identical to the original theory with N reduced by one (and F unchanged); the second has a $U(1)$ gauge group with $2F$ pairs of oppositely charged matter fields; the third is a single free chiral superfield. There is no superpotential in this description.

In the first dual description, this vacuum corresponds to $\langle A_1 \rangle \neq 0$, which breaks $SO(2N+5) \to SO(2N+3) \times U(1)$. It is easy to check that the low-energy theory again consists of three sectors: the first is exactly the dual description of the Sp sector described above, and the second and third are identical to the corresponding sectors above. Note that in this dual, the physics of this vacuum is described by spontaneous breaking of the gauge group in both the original and the dual description.

One can consider more complicated vacua where $\langle A \rangle$ has more non-zero eigenvalues by iterating the analysis above. One might worry about the fact that the field A_1 in the dual description apparently has $N+2$ degrees of freedom along the D-flat direction, while the field A in the original description has only N degrees of freedom. However, it is easy to see that giving non-zero vacuum expectation values to N components of A_1 in the dual leads to a confining theory which does not have any additional flat directions corresponding to adjoint VEVs; the apparent extra flat directions of the dual have been lifted by strong gauge dynamics.

As another example, consider a direction in moduli space corresponding to $\langle Q\rangle \neq 0$, $\langle A\rangle = 0$ in the original description of the theory. Imposing the D-flatness condition, the simplest possibility is

$$\langle Q\rangle = \begin{pmatrix} a1_2 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}. \tag{A.5}$$

This breaks the gauge symmetry $Sp(2N) \to Sp(2N-2)$, and the massless fields decompose under $Sp(2N-2)$ as an adjoint, $2F$ fundamentals, and $4F$ singlets. There is no superpotential.

In the first dual description, this vacuum corresponds to $\langle M_0\rangle \neq 0$. This breaks the flavor symmetry $SU(2F) \to SU(2F-2)\times SU(2)$, and some of the fields $\widetilde{Q}$ become massive. The gauge group is not broken in this theory, but the theory is more strongly coupled because there are fewer matter fields. This description is not obviously dual to the one discussed in the previous paragraph. To see that they are equivalent, take the dual of the $Sp(2F+2)$ gauge group in this description. We then obtain a theory that is similar to the deconfined description of the theory. The SO gauge group of this description is confining, and writing the low-energy theory of the SO mesons we recover the description above.

We can analyze more complicated vacuum expectation values for Q by iterating the above analysis. We therefore have a consistency check that can be written diagrammatically as

$$\begin{array}{ccccc} \text{original} & \to & \text{deconfined} & \to & \text{dual} \\ \downarrow & & & & \downarrow \\ \text{original with VEV} & \leftarrow & \text{deconfined with VEV} & \leftarrow & \text{dual with VEV} \end{array} \tag{A.6}$$

where the horizontal arrows denote duality or (de)confinement transformations, and the vertical arrows denote taking VEVs corresponding to a given direction in moduli space.

A.4 $N = 0$

An amusing consistency check is to consider the theory with $N = 0$. In this case the original theory is trivial, but our dual descriptions appear at first sight to be non-trivial. Consider the first dual. In this theory, the Sp group has the right number of flavors to confine without breaking chiral symmetry [3, 6]. The fields $\widetilde{Q}$, $\widetilde{x}_1$, and $\widetilde{p}_2$ confine into mesons that combine with M_0, A_1, m_1, and m_2 to become massive. This leaves an effective theory with the field p_3 and a meson field $M = \widetilde{x}_1\widetilde{p}_2$, with superpotential

$$W = p_3 M^2. \tag{A.7}$$

p_3 is a singlet under $SO(5)$, while M transforms in the defining representation. An $SO(5)$ gauge theory with one flavor confines without breaking chiral symmetry [5], so the the low-energy theory can be written in terms of the composite meson $N = M^2$. The superpotential then gives a mass to N and p_3, leaving a low-energy theory with no massless degrees of freedom. This is exactly what is required for consistency with the original theory.

B Deconfining Arbitrary 2-Index Tensors

The methods we have used can be extended to write dual descriptions of any gauge theory with *SU*, *SO*, or *Sp* gauge groups containing at most 2-index tensor representations. Note that supersymmetric gauge theories with matter in 3-index tensor representations are not asymptotically free for large N (specifically, $N > 5$ for $Sp(2N)$, $N > 8$ for $SO(N)$, and $N > 12$ for $SU(N)$).

In this sense, these methods allow us to construct dual descriptions of "almost all" supersymmetric gauge theories with tensor representations.

As discussed in the main text, the idea is to "deconfine" all the 2-index tensors by introducing new confining gauge interactions whose low-energy dynamics is a theory of mesons. The simplest approach is to introduce only those fields required to produce the 2-index tensor as a bound state, but then the number of anomaly free $U(1)$ symmetries do not match because there is an extra anomaly cancellation constraint from the confining gauge group. Also, if the confining gauge group is *Sp* or *SU*, the mesons have an unwanted dynamical superpotential. These problems are solved simultaneously by adding additional fields that are fundamentals under the confining gauge group, together with some singlets and a superpotential to make the additional mesons massive. The result is a gauge theory containing only fundamental representations of all gauge groups, and one can apply known dualities to obtain dual descriptions from this.

For an antisymmetric 2-index tensor X^{ab} with $a, b = 1, \ldots, N$ transforming under some gauge group G, one introduces an additional $Sp(2N')$ gauge group with matter fields

$$(x)^{aa'}, \quad (p_1)^{a'j}, \quad j = 1, \ldots, K \tag{B.1}$$

for some K. In addition, one introduces $Sp(2N')$ singlets

$$(p_2)_{a,j}\,, \quad (p_3)_{jk} \tag{B.2}$$

with superpotential couplings

$$\delta W = x p_1 p_2 + p_1 p_1 p_3, \tag{B.3}$$

where all indices are contracted in the obvious way. The fields with their transformation properties are displayed in Table 7.

field	G	$Sp(2N')$	$SU(K)$
X	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$		
x	$\Box$	$\Box$	$\mathbf{1}$
p_1	$\mathbf{1}$	$\Box$	$\Box$
p_2	$\overline{\Box}$	$\mathbf{1}$	$\overline{\Box}$
p_3	$\mathbf{1}$	$\mathbf{1}$	$\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}$

Table 7: Field content of the deconfinement module for an antisymmetric tensor.

If we take $N' = \frac{1}{2}(N + K) - 2$ (choosing K so that N' is an integer) then this theory confines and gives rise to a low-energy theory with of a single meson field, which can be identified as

$$X^{ab} = \epsilon_{a'b'}(x)^{aa'}(x)^{bb'}. \tag{B.4}$$

Note that for $K > 1$, there is an additional global $SU(K)$ symmetry in the ultraviolet, but the only fields that transform under this symmetry are not present in the low-energy theory. We can now write a dual description by applying the known duality for theories with matter only in the fundamental representation to the group G.

Symmetric 2-index tensors are treated in detail in Section 3, and so the only case left to discuss is an adjoint $X^a{}_b$ of $SU(N)$. (Adjoint representations of SO are antisymmetric tensors.) We "deconfine" the adjoint by introducing a new $SU(N')$ gauge group with matter fields

$$(x)^{aa'}, \quad (\bar{x})_{aa'}, \quad (p_1)^{a'j}, \quad (\bar{p}_1)_{a'j'} \;. \tag{B.5}$$

(Note the bar does not indicate complex conjugation.) If we choose $N' = N + K - 1$, then this theory confines and gives rise to a low-energy effective theory consisting of composite mesons and baryons and a non-perturbative superpotential. To eliminate the unwanted states and the non-perturbative superpotential we add the following fields

$$p_2, \quad (p_3)^{j'}_a, \quad (\bar{p}_3)^a_j, \quad (p_4)^{j'}_j, \quad (p_5)^a, \quad (\bar{p}_5)_a, \quad (p_6)^j, \quad (\bar{p}_6)_{j'}, \tag{B.6}$$

and tree level superpotential

$$\begin{aligned}\delta W \;=\;& p_2 x\bar{x} + p_3 x\bar{p}_1 + \bar{p}_3\bar{x}p_1 + p_4 p_1\bar{p}_1 \\ &+ p_5(x)^{N-1}(p_1)^K + \bar{p}_5(\bar{x})^{N-1}(\bar{p}_1)^K + p_6(x)^N(p_1)^{K-1} + \bar{p}_6(\bar{x})^N(\bar{p}_1)^{K-1}.\end{aligned} \tag{B.7}$$

The term $p_2 x\bar{x}$ eliminates the trace of the meson field $(x\bar{x})^a{}_b$ as a dynamical field at low energies. The fields p_3, $\bar{p}_3$, p_4 eliminate other unwanted mesons, and p_5, $\bar{p}_5$, p_6, $\bar{p}_6$ eliminate the baryons from the low energy spectrum. The only massless degree of

freedom left is the composite field $X^a{}_b$ (with no superpotential), as desired. We can now write a dual description by applying the duality of Seiberg to the gauge group corresponding to the indices $a, b, \ldots$. In this way, we can write dual descriptions for the $SU(N)$ Kutasov–Schwimmer model (with no tree-level superpotential) similar to the ones constructed above in the $Sp(2N)$ case. The analysis of these duals procedes in direct analogy with that in the main body of the paper, and will not be given here.

field	$SU(N)$	$SU(N')$	$SU(K)$	$SU(K)'$
X	$\mathbf{A}$			
x	$\square$	$\square$	$\mathbf{1}$	$\mathbf{1}$
$\bar{x}$	$\overline{\square}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$
p_1	$\mathbf{1}$	$\square$	$\square$	$\mathbf{1}$
$\bar{p}_1$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$
p_2	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
p_3	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\square$
$\bar{p}_3$	$\square$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
p_4	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$	$\square$
p_5	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\bar{p}_5$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
p_6	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$
$\bar{p}_6$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$

Table 8: Field content of the deconfinement module for an $SU(N)$ adjoint.

References

[1] N. Seiberg, *Phys. Lett.* **318B**, 469 (1993), `hep-th/9408013`.

[2] N. Seiberg, *Phys. Rev.* **D49**, 6857 (1994), `hep-th/9402044`.

[3] N. Seiberg, *Nucl. Phys.* **B435**, 129 (1995), `hep-th/9411149`.

[4] For a recent review see K. Intriligator and N. Seiberg, RU-95-48, IASSNS-HEP-95/70, `hep-th/9509066`.

[5] K. Intriligator and N. Seiberg, *Nucl. Phys.* **B444**, 125 (1995), `hep-th/9503179`.

[6] K. Intriligator and P. Pouliot, *Phys. Lett.* **353B**, 471 (1995), `hep-th/9505006`.

[7] K. Intriligator, R.G. Leigh, and M.J. Strassler, RU-95-38, `hep-th/9506148`.

[8] M. Berkooz, *Nucl. Phys.* **B452**, 513 (1995), `hep-th/9505067`; for further applications see also: ref. [7]; ref. [9]; P. Pouliot and M.J. Strassler, *Phys. Lett.* **370B**, 76 (1996), `hep-th/9510228`.

[9] P. Pouliot, *Phys. Lett.* **367B**, 151 (1996), `hep-th/9510148`.

[10] D. Kutasov, *Phys. Lett.* **351B**, 230 (1995), `hep-th/9503086`.

[11] D. Kutasov and A. Schwimmer, *Phys. Lett.* **354B**, 315 (1995), `hep-th/9505004`.

[12] R.G. Leigh and M.J. Strassler, *Phys. Lett.* **356B**, 492 (1995), `hep-th/9505088`.

[13] D. Kutasov, A. Schwimmer, and N. Seiberg, *Nucl. Phys.* **B459**, 455 (1996), `hep-th/9510222`.

[14] S. Elitzur, A. Forge, A. Giveon, and E. Rabinovici, *Nucl. Phys.* **B459**, 160 (1996), `hep-th/9509130`.

[15] N. Seiberg and E. Witten, *Nucl. Phys.* **B431**, 484 (1994), hep-th/9408099.

[16] K. Intriligator, *Nucl. Phys.* **B448**, 187 (1995), `hep-th/9505051`.

[17] M.A. Luty and W. Taylor IV, *Phys. Rev.* **D53**, 3399 (1996), `hep-th/9506098`.

[18] V.A. Novikov, M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, *Nucl. Phys.* **B229**, 1983 (381); M.A. Shifman and A.I. Vainshtein, *Nucl. Phys.* **B277**, 1986 (456); *Nucl. Phys.* **B359**, 1991 (571).

[19] T. Banks and A. Zaks, *Nucl. Phys.* **B196**, 189 (1982).

[20] G. Mack, *Comm. Math. Phys.* **55**, 1 (1977); M. Flato and C. Fronsdal, *Lett. Math. Phys.* **8**, 159 (1984); V.K. Dobrev and V.B. Petkova, *Phys. Lett.* **162B**, 127 (1985).